



Advanced Variance Analysis Of Functions Containing Quotients: Avoiding Information Loss From Treating Ratios As A Single Variable

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ABSTRACT

Variance analysis calculations are commonly performed on a two-variable product. In cases of the original function being a three-variable quotient, often the first step is to divide one variable in the numerator by another variable in the denominator, resulting in one new, aggregate variable replacing the original two. Such reduction to a two-variable function leads to a loss of information. This article presents models designed to avoid the aggregation of variables, to be used with only original variables. The models incorporate the concept of responsibility centers, allowing for a more appropriate set of variance analysis calculations for the firm.

INTRODUCTION

Variance analysis (VA) is most commonly performed on a two-variable product, such as *Revenue = Unit Price × Number of Units*, and cost accounting textbooks tend to provide only a single VA model to perform the necessary calculations (Blocher et al, 2022). Provided the variables are not functions of other variables (e.g., *Earnings per Share* being treated as a single variable even though it is calculated as *Total Earnings/Number of Shares*), and there is no reason to view them differently (e.g., one is considered more important), the textbook model suffices. However, often in practice at least one of these simplifying assumptions will not apply.

First, there are numerous examples of functions that appear to be a product of two original variables, but in fact include at least one aggregate variable that itself is a function of two or more original variables. Common examples include efficiency and conversion ratios. While accounting and finance professionals make extensive use of ratios, performing VA on functions that include quotients seems to be difficult for one reason or another. In fact, some works on VA specifically attempt to avoid ratios by transforming them into differences. Diewert (2000 and 2005) acknowledge the importance of VA of ratios, and at the same time, “the ratio approach is not one that the business and accounting community finds natural.” The author provides methods for performing VA on functions that include quotients by treating them as differences instead.

As an example, consider labor efficiency, where *Capacity Cost Rate* is calculated from two, original variables as *Compensation/Capacity*, and is then treated as a single variable in the function *Cost = Time × Capacity Cost Rate*. Calculating *Time* variance and *Capacity Cost Rate* variance using the

standard textbook model is straightforward at this point. However, by aggregating two original variables into a quotient that is treated as a single variable, even determining the sign and the magnitude of the two original variables’ variance can become problematic, if not impossible. With respect to sign, given the additivity property of the variances (i.e., the variance of a quotient equals the sum of each variable’s variance), there are five combinations of numerator variable’s variance sign and denominator variable’s variance sign that could result in a negative quotient variance (-/-, -/0, 0/-, -/+ and +/-). Similarly, there are five original variable variance sign combinations that can result in a positive quotient variance (+/+, +/0, 0/+, -/+ and +/-). As a result, if the quotient variance is negative (positive), the only insight into the sign of the variance of the original variables one can confidently infer is that at least one variance is negative (positive). With respect to magnitude, there are an infinite number of combinations of numerator variance size and denominator variance size that can result in a given quotient variance. As a result, no insight into the magnitude of the variance of the two original variables can be gained by only considering the quotient variance. For a specific example with a numerical demonstration, see Milani and Perri (2013), who study average cost per hamburger in a restaurant setting.

Second, the standard textbook VA model for a two-variable product, $f(x, y) = xy$, is: x variance = Δxy_a and y variance = Δyx_b , where the difference and average of any variable u is $\Delta u = u_a - u_b$, and $\bar{u} = (u_a + u_b)/2$, respectively. A budget value is indicated by subscript b and an actual value by subscript a . In textbooks, noticeably absent from the presentation of the model is a discussion as to which variable in the function of interest should be denoted by x and which by y . This distinction matters. For VA of the simple revenue function, is *Unit Price* variance calculated

using *Actual Units Sold* or *Budget Units Sold*? If a given variable is considered qualitatively different than the other (e.g., more important), this may provide justification for denoting it as x instead of y . To address this issue in the current article, the concept of a responsibility center is incorporated (Sorochuk et al, 2023a and 2023b).

The purpose of this article is to provide new VA models specifically designed to avoid the loss of information resulting from aggregating original variables (specifically, as a quotient) into a new variable, that itself is treated as a single variable when performing VA. The models include two- and three-variable functions of only original, non-aggregate variables, whose individual variances can be added if the variance of an aggregate variable is desired. Models for every applicable combination and number of responsibility centers are provided. A numerical example with three original variables demonstrates the shortcomings of aggregating variables before performing VA, and the superior results achieved by using the new models.

MODELS

The VA models here are generated based on the concept of responsibility centers. A responsibility center is an entity (person, department, etc.) responsible for influencing change on the value of its respective variable. For example, if a firm has the ability to set prices for the product it is selling, it will likely have a pricing or marketing department responsible for that function. The pricing or marketing department would be the responsibility center. Conversely, if the firm is selling a pure commodity that has a spot price set by the open market (e.g., gold, corn, etc.) there would likely not be a person nor department inside the firm being held accountable for the selling price of the commodity. No responsibility center for price would exist.

The concept of responsibility centers is incorporated as follows. Kaplan (2004) notes, "... managers expect that many of their indirect and support expenses should be managed or controlled based on *actual* activity levels during the period." Considering this, in generating the models, the variance of a variable that does not belong to a responsibility center is evaluated using the budget value of any variables that do belong to a responsibility center. Conversely, the variance of a variable that does belong to a responsibility center is evaluated using the actual value of any variables that do not belong to a responsibility center.

Complete details on generating the models (including the algorithm and an example) can be found at Sorochuk (2023). To account for all possible numbers and combinations of responsibility centers, a given n -variable function will have 0, 1, 2, ..., n possible responsibility centers (zero or one for each variable), and $2^n - 1$ models, as for a given function, the zero- and n -responsibility center models are the same. Models for 1) a two-variable quotient, 2) a three-variable quotient with a two-variable denominator, and 3) a three-variable quotient with a two-variable numerator are shown in Tables 1-3, respectively. As a matter of notation, variables are indicated by an italicized lower-case letter or word and belong to the responsibility center indicated by the respective upper-case letter or word, if at all.

An example of a two-variable quotient is a simple, manufacturing efficiency ratio: *Units of Output / Total Machine Hours*. The applicable models are shown in Table 1, with x = *Units of Output* and y = *Total Machine Hours*.

TABLE 1. Variance Models for a Two-Variable Quotient

Variance	Responsibility Center(s)		
	X (Model 1)	Y (Model 2)	X and Y* (Model 3)
x	$\Delta x \frac{1}{y_a}$	$\Delta x \frac{1}{y_b}$	$\Delta x \left(\frac{1}{y} \right)$
y	$\left(\Delta \frac{1}{y} \right) x_b$	$\left(\Delta \frac{1}{y} \right) x_a$	$\left(\Delta \frac{1}{y} \right) \bar{x}$
Total		$\frac{x_a}{y_a} - \frac{x_b}{y_b}$	

*Also applicable for no responsibility centers.

An example of a three-variable quotient with a two-variable denominator, is dividend yield ratio, often presented as a two-variable quotient:

$$\text{Dividend Yield Ratio} = \frac{\text{Dividend per Share}}{\text{Share Price}}. \quad (1)$$

However, expanding *Dividend per Share* into a two-variable quotient (*Total Dividends / Number of Shares*) and rewriting gives:

$$\text{Dividend Yield Ratio} = \frac{\text{Total Dividends}}{\text{Number of Shares} \times \text{Share Price}}. \quad (2)$$

The three-variable form is particularly useful for a firm whose total dividend payout changed drastically (perhaps due to declined earnings), or number of outstanding shares changed, (perhaps due to a buyback program). The

applicable models are shown in Table 2, with $x = \text{Total Dividends}$, $y = \text{Number of Shares}$, and $z = \text{Share Price}$.

An example of a three-variable quotient with a two-variable numerator is labor efficiency, where cost is initially calculated as a two-variable product:

$$\text{Cost} = \text{Time per Task} \times \text{Capacity Cost Rate}. \quad (3)$$

Expanding *Capacity Cost Rate* to a two-variable quotient (*Compensation / Time Available*) and rewriting gives:

$$\text{Cost} = \frac{\text{Time per Task}^1 \times \text{Compensation}}{\text{Time Available}}. \quad (4)$$

The three-variable form is particularly useful for a firm that has recently changed an employee's compensation (perhaps due to a contract negotiation) or amount of time to work on a task. The applicable models are shown in Table 3, with $x = \text{Time per Task}$, $y = \text{Compensation}$, and $z = \text{Time Available}$. A numerical example with discussion follow.

TABLE 2. Variance Models for a Three-Variable Quotient with a Two-Variable Denominator

Variance	Responsibility Center(s)		
	X (Model 4)	Y (Model 5)	Z (Model 6)
x	$\Delta x \frac{1}{y_a} \frac{1}{z_a}$	$\Delta x \left(\frac{1}{y_b} \left(\frac{1}{z} \right) \right)$	$\Delta x \left(\left(\frac{1}{y} \right) \frac{1}{z_b} \right)$
y	$\left(\Delta \frac{1}{y} \right) \left(x_b \left(\frac{1}{z} \right) \right)$	$\left(\Delta \frac{1}{y} \right) \left(x_a \frac{1}{z_a} \right)$	$\left(\Delta \frac{1}{y} \right) \left(\bar{x} \frac{1}{z_b} \right)$
z	$\left(\Delta \frac{1}{z} \right) \left(x_b \left(\frac{1}{y} \right) \right)$	$\left(\Delta \frac{1}{z} \right) \left(\bar{x} \frac{1}{y_b} \right)$	$\left(\Delta \frac{1}{z} \right) \left(x_a \frac{1}{y_a} \right)$
Total	$\frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b}$		

Variance	Responsibility Center(s)			
	X and Y (Model 7)	X and Z (Model 8)	Y and Z (Model 9)	X, Y and Z* (Model 10)
x	$\Delta x \left(\left(\frac{1}{y} \right) \frac{1}{z_a} \right)$	$\Delta x \left(\frac{1}{y_a} \left(\frac{1}{z} \right) \right)$	$\Delta x \left(\frac{1}{y_b} \frac{1}{z_b} \right)$	$\Delta x \left(\left(\frac{1}{y} \right) \left(\frac{1}{z} \right) + \frac{\Delta \frac{1}{y} \Delta \frac{1}{z}}{12} \right)$
y	$\left(\Delta \frac{1}{y} \right) \left(\bar{x} \frac{1}{z_a} \right)$	$\left(\Delta \frac{1}{y} \right) \left(x_b \frac{1}{z_b} \right)$	$\left(\Delta \frac{1}{y} \right) \left(x_a \left(\frac{1}{z} \right) \right)$	$\left(\Delta \frac{1}{y} \right) \left(\bar{x} \left(\frac{1}{z} \right) + \frac{\Delta x \Delta \frac{1}{z}}{12} \right)$
z	$\left(\Delta \frac{1}{z} \right) \left(x_b \frac{1}{y_b} \right)$	$\left(\Delta \frac{1}{z} \right) \left(\bar{x} \frac{1}{y_a} \right)$	$\left(\Delta \frac{1}{z} \right) \left(x_a \left(\frac{1}{y} \right) \right)$	$\left(\Delta \frac{1}{z} \right) \left(\bar{x} \left(\frac{1}{y} \right) + \frac{\Delta x \Delta \frac{1}{y}}{12} \right)$
Total	$\frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b}$			

*Also applicable for no responsibility centers.

¹ A possible next step is to also expand *Time per Task* into a two-variable quotient, *Total Time Spent on Tasks / Number of Tasks*, creating a four-variable quotient cost function.

TABLE 3. Variance Models for a Three-Variable Quotient with a Two-Variable Numerator

Variance	Responsibility Center(s)		
	X (Model 11)	Y (Model 12)	Z (Model 13)
x	$\Delta x y_a \frac{1}{z_a}$	$\Delta x y_b \left(\frac{1}{z} \right)$	$\Delta x \bar{y} \frac{1}{z_b}$
y	$\Delta y x_b \left(\frac{1}{z} \right)$	$\Delta y x_a \frac{1}{z_a}$	$\Delta y \bar{x} \frac{1}{z_b}$
z	$\left(\Delta \frac{1}{z} \right) x_b \bar{y}$	$\left(\Delta \frac{1}{z} \right) \bar{x} y_b$	$\left(\Delta \frac{1}{z} \right) x_a y_a$
Total	$\frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b}$		

Variance	Responsibility Center(s)			
	X and Y (Model 14)	X and Z (Model 15)	Y and Z (Model 16)	X, Y and Z* (Model 17)
x	$\Delta x \bar{y} \frac{1}{z_a}$	$\Delta x y_a \left(\frac{1}{z} \right)$	$\Delta x y_b \frac{1}{z_b}$	$\Delta x \left(\bar{y} \left(\frac{1}{z} \right) + \frac{\Delta y \Delta \frac{1}{z}}{12} \right)$
y	$\Delta y \bar{x} \frac{1}{z_a}$	$\Delta y x_b \frac{1}{z_b}$	$\Delta y x_a \left(\frac{1}{z} \right)$	$\Delta y \left(\bar{x} \left(\frac{1}{z} \right) + \frac{\Delta x \Delta \frac{1}{z}}{12} \right)$
z	$\left(\Delta \frac{1}{z} \right) x_b y_b$	$\left(\Delta \frac{1}{z} \right) \bar{x} y_a$	$\left(\Delta \frac{1}{z} \right) x_a \bar{y}$	$\left(\Delta \frac{1}{z} \right) \left(\bar{x} \bar{y} + \frac{\Delta x \Delta y}{12} \right)$
Total	$\frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b}$			

*Also applicable for no responsibility centers.

NUMERICAL EXAMPLE

This section presents a numerical example of the labor efficiency example with a three-variable quotient as a cost function. The goal is to demonstrate the use of the new models and compare the results from using an aggregate variable vs. using only original variables. The variables and parameters are shown in Table 4. Details of the calculations and a discussion follow.

TABLE 4. Example Variables and Parameters

Variable	Budget	Actual
Time Required per Task, <i>time</i> (min)	100	150
Annual Compensation, <i>comp</i> (\$)	\$360,000.00	\$240,000.00
Annual Capacity Available to Perform Tasks, <i>cap</i> (min)	90,000	80,000
Capacity Cost Rate, <i>ccr</i> = <i>comp/cap</i> (\$/min)	\$4.00	\$3.00

The annual cost of working on tasks is calculated as

$$Cost = \frac{time \times comp}{cap}, \quad (5)$$

a function of three original variables. If the analyst is interested in reducing the number of variables to be able to use the traditional, two-variable product model, *comp* and *cap* are aggregated into a single variable, capacity cost rate, *ccr*, as

$$ccr = \frac{comp}{cap}. \quad (6)$$

With this new variable, *Cost* can be expressed as a two-variable product,

$$Cost = time \times ccr, \quad (7)$$

a function of one original variable and one aggregate variable.

Following aggregation, VA is performed on Equation 7 using two-variable product models shown in Table 5 (Sorochuk 2023a and 2023b) with $x = \text{time}$ and $y = \text{ccr}$. These models are analogous to those found in Table 1, but generated for a product instead of a quotient. The results are shown in Table 6.

TABLE 5. Variance Models for a Two-Variable Product

Variance	Responsibility Center(s)		
	X (Model 18)	Y (Model 19)	X and Y* (Model 20)
x	Δxy_a	Δxy_b	$\Delta x\bar{y}$
y	Δyx_b	Δyx_a	$\Delta y\bar{x}$
Total	$x_a y_a - x_b y_b$		

*Also applicable for no responsibility centers.

TABLE 6. Example Results Using Cost Modeled as a Two-Variable Product

Variance	Responsibility Center(s)		
	TIME (Model 18)	CCR (Model 19)	TIME and CCR* (Model 20)
<i>time</i>	\$150.00	\$200.00	\$175.00
<i>ccr</i>	(\$100.00)	(\$150.00)	(\$125.00)
<i>Cost</i>	\$50.00		

*Also applicable for no responsibility centers.

Without aggregation, VA is performed on Equation 5 using the models in Table 3, with $x = \text{time}$, $y = \text{comp}$, and $z = \text{cap}$. From the additivity property of variances, *ccr* variance is calculated as the sum of *comp* variance and *cap* variance. The results are shown in Table 7.

In comparing the results, there are two obvious differences related to the different number of variables of each function; the number of models applicable for each function, and the number of variances that can be calculated for each.

TABLE 7. Example Results Using Cost Modeled as a Three-Variable Quotient with a Two-Variable Numerator

Variance	Responsibility Center(s)		
	TIME (Model 11)	COMP (Model 12)	CAP (Model 13)
<i>time</i>	\$150.00	\$212.50	\$166.67
<i>comp</i>	(\$141.67)	(\$225.00)	(\$166.67)
<i>cap</i>	\$41.67	\$62.50	\$50.00
<i>ccr</i>	(\$100.00)	(\$162.50)	(\$116.67)
<i>Cost</i>	\$50.00		

Variance	Responsibility Center(s)			
	TIME and COMP (Model 14)	TIME and CAP (Model 15)	COMP and CAP (Model 16)	TIME, COMP and CAP* (Model 17)
<i>time</i>	\$187.50	\$141.67	\$200.00	\$176.39
<i>comp</i>	(\$187.50)	(\$133.33)	(\$212.50)	(\$177.78)
<i>cap</i>	\$50.00	\$41.67	\$62.50	\$51.39
<i>ccr</i>	(\$137.50)	(\$91.67)	(\$150.00)	(\$126.39)
<i>Cost</i>	\$50.00			

*Also applicable for zero responsibility centers.

First, for the two-variable product function there are only three ($= 2^2 - 1$) models, i.e., combinations of (non)responsibility centers (i.e., models), whereas for the three-variable quotient function there are seven ($= 2^3 - 1$). This difference results not only from the number of variables in each function, but the number of values each variable can take on. The models allow for each variable to take on only two values, actual and budget. For the original variables, this is satisfactory, as each has only actual and budget values to begin with. However, this is

problematic for the aggregate variable, *ccr*. An aggregate of two original variables, each allowed to take on two values, should result in $2 \times 2 = 4$ combinations of those two values. However, by definition, $ccr_a = comp_a / cap_a$ and $ccr_b = comp_b / cap_b$. This highlights a significant problem resulting from the aggregation of the two original variables: when calculating an allowed value of *ccr*, it is not possible to use the budget value of one of *comp* or *cap*, and the actual value of the other. This restriction contributes to the reduced number of

models that can be generated with an aggregate variable. Fortunately, Models 12 through 15 eliminate the problem of the reduced number of values that *ccr* can take on as an aggregate variable in the two-variable function. These four models account for the ways that one of *comp* and *cap* will be assigned to a responsibility center and the other will not.

Second, by definition, the two-variable product function will have two variances, and the three-variable quotient function will have three. In fact, with minimal additional effort, *ccr* variance can be calculated for the three-variable quotient function by adding *comp* variance and *cap* variance, values already calculated using the models in Table 3.

It is encouraging to note that some results are the same in the two sets of models. Given that *time* is the only variable in common between the two functions, it is meaningful to compare the results of Model 11 with Model 18 (*time* is the only variable belonging to a responsibility center) as well as Model 16 with Model 19 (*time* is the only variable not belonging to a responsibility center). In both comparisons, *time* variance and *ccr* variance are the same. In addition, although not exactly the same, those two variances are in very close agreement when comparing the results of Models 17 and 20, both of which have all or none of the variables assigned to a responsibility center. This agreement supports the validity of the new models, assuming the simpler, two-variable models are correct to begin with.

FUTURE WORK

This article provides models for VA of two and three-variable quotients, however more complicated functions can easily be found. As mentioned above, the cost associated with labor efficiency can be expressed as a four-variable quotient. For an even more sophisticated example, consider sales mix for a multinational firm selling multiple products (denoted by subscripts $i = 1, 2, \dots$). The margin contributing to a given sales mix, *SM*, for product i is calculated as:

$$SM_i = s_i q_i m_i x_i, \quad (8)$$

where

s denotes sales mix percentage,

q denotes quantity sold,

m denotes unit margin, and

x denotes the margin exchange rate.

When SM_i is expressed as in Equation 8, an assumption is that unit price p and unit cost c used to calculate unit margin ($m = p - c$) are in the same currency. If this is not the case, the required foreign exchange rates for product i are denoted as x_i^p and x_i^c , respectively. Note also that s_i is calculated as $q_i / \sum_i q_i$. Substituting into Equation 8 gives:

$$SM_i = \frac{q_i}{\sum_i q_i} q_i (x_i^p p_i - x_i^c c_i) \quad (9)$$

When expanded, the right-hand side of Equation 9 becomes a function of two terms with five factors each (seven factors in total), as q_i^2 is treated as two factors and the denominator $\sum_i q_i$ as another. The VA models for such a function can be generated using the algorithm in Sorochuk (2023). Considering all the possible combinations of (non-)responsibility centers, this would require considerable effort, as there would be $2^5 - 1 = 31$ models for each five-factor term.

CONCLUSION

When performing VA of a function that includes a quotient of two original variables, common practice is to evaluate the quotient and treat the result as a single variable, to then be used in VA. Such aggregation results in a loss of information, and virtually no insight can be gained regarding the variance of the original two variables, only the newly-created aggregate variable. This article provides new models generated specifically to avoid information loss, that consider only the original variables as well as the number and combination of responsibility centers. These new models allow for the variance of each and every original variable to be calculated in a single step, and with minimal additional effort to calculate the variance of an aggregate variable.

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APPENDIX - PROOFS

This appendix contains mathematical proofs that each of the 17 models generated by the algorithm (Sorochuk, 2023) is, a correct way of expressing the respective difference of quotients.

Two-variable quotient with X responsibility center (Model 1)

$$\begin{aligned}\frac{x_a}{y_a} - \frac{x_b}{y_b} &= \Delta x \frac{1}{y_a} + \left(\Delta \frac{1}{y}\right) x_b \\ &= (x_a - x_b) \frac{1}{y_a} + \left(\frac{1}{y_a} - \frac{1}{y_b}\right) x_b \\ &= \frac{x_a}{y_a} - \frac{x_b}{y_a} + \frac{x_b}{y_a} - \frac{x_b}{y_b} \\ &= \frac{x_a}{y_a} - \frac{x_b}{y_b} \quad \square\end{aligned}$$

Two-variable quotient with Y responsibility center (Model 2)

$$\begin{aligned}\frac{x_a}{y_a} - \frac{x_b}{y_b} &= \Delta x \frac{1}{y_b} + \left(\Delta \frac{1}{y}\right) x_a \\ &= (x_a - x_b) \frac{1}{y_b} + \left(\frac{1}{y_a} - \frac{1}{y_b}\right) x_a \\ &= \frac{x_a}{y_b} - \frac{x_b}{y_b} + \frac{x_a}{y_a} - \frac{x_a}{y_b} \\ &= \frac{x_a}{y_a} - \frac{x_b}{y_b} \quad \square\end{aligned}$$

Two-variable quotient with both X and Y (or no) responsibility centers (Model 3)

$$\begin{aligned}\frac{x_a}{y_a} - \frac{x_b}{y_b} &= \Delta x \left(\frac{1}{y}\right) + \left(\Delta \frac{1}{y}\right) \bar{x} \\ &= (x_a - x_b) \frac{\left(\frac{1}{y_a} + \frac{1}{y_b}\right)}{2} \\ &\quad + \left(\frac{1}{y_a} - \frac{1}{y_b}\right) \frac{(x_a + x_b)}{2} \\ &= \frac{\frac{x_a}{y_a} + \frac{x_a}{y_b} - \frac{x_b}{y_a} - \frac{x_b}{y_b} + \frac{x_a}{y_a} + \frac{x_b}{y_a} - \frac{x_a}{y_b} - \frac{x_b}{y_b}}{2} \\ &= \frac{2 \frac{x_a}{y_a} - 2 \frac{x_b}{y_b}}{2} \\ &= \frac{x_a}{y_a} - \frac{x_b}{y_b} \quad \square\end{aligned}$$

Three-variable quotient with two-variable denominator and X responsibility center (Model 4)

$$\begin{aligned}\frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b} &= \Delta x \frac{1}{y_a z_a} + \left(\Delta \frac{1}{y}\right) \left(x_b \frac{1}{z}\right) \\ &\quad + \left(\Delta \frac{1}{z}\right) \left(x_b \frac{1}{y}\right) \\ &= (x_a - x_b) \frac{1}{y_a z_a} \\ &\quad + \left(\frac{1}{y_a} - \frac{1}{y_b}\right) \left(x_b \frac{\left(\frac{1}{z_a} + \frac{1}{z_b}\right)}{2}\right)\end{aligned}$$

$$\begin{aligned}&+ \left(\frac{1}{z_a} - \frac{1}{z_b}\right) \left(x_b \frac{\left(\frac{1}{y_a} + \frac{1}{y_b}\right)}{2}\right) \\ &= \frac{x_a}{y_a z_a} - \frac{x_b}{y_a z_a} + \frac{\frac{x_b}{y_a z_a} - \frac{x_b}{y_a z_b} + \frac{x_b}{y_b z_a} - \frac{x_b}{y_b z_b}}{2} \\ &= \frac{\frac{2x_a}{y_a z_a} - \frac{2x_b}{y_a z_a} + \frac{x_b}{y_a z_a} - \frac{x_b}{y_b z_a} + \frac{x_b}{y_a z_b} - \frac{x_b}{y_b z_b}}{2} \\ &\quad + \frac{\frac{x_b}{y_a z_a} - \frac{x_b}{y_a z_b} + \frac{x_b}{y_b z_a} - \frac{x_b}{y_b z_b}}{2} \\ &= \frac{\frac{2x_a}{y_a z_a} - \frac{2x_b}{y_b z_b}}{2} \\ &= \frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b} \quad \square\end{aligned}$$

Three-variable quotient with two-variable denominator and Y responsibility center (Model 5)

$$\begin{aligned}\frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b} &= \Delta x \left(\frac{1}{y_b} \frac{1}{z}\right) + \left(\Delta \frac{1}{y}\right) \left(x_a \frac{1}{z_a}\right) \\ &\quad + \left(\Delta \frac{1}{z}\right) \left(\bar{x} \frac{1}{y_b}\right) \\ &= (x_a - x_b) \left(\frac{1}{y_b} \frac{\left(\frac{1}{z_a} + \frac{1}{z_b}\right)}{2}\right) \\ &\quad + \left(\frac{1}{y_a} - \frac{1}{y_b}\right) \left(x_a \frac{1}{z_a}\right) \\ &\quad + \left(\frac{1}{z_a} - \frac{1}{z_b}\right) \left(\frac{(x_a + x_b)}{2} \frac{1}{y_b}\right) \\ &= \frac{\frac{x_a}{y_b z_a} - \frac{x_b}{y_b z_a} + \frac{x_a}{y_b z_b} - \frac{x_b}{y_b z_b}}{2} \\ &\quad + \frac{\frac{x_a}{y_a z_a} - \frac{x_a}{y_b z_a} + \frac{x_a}{y_b z_a} - \frac{x_a}{y_b z_b} + \frac{x_b}{y_b z_a} - \frac{x_b}{y_b z_b}}{2} \\ &= \frac{\frac{x_a}{y_b z_a} - \frac{x_b}{y_b z_a} + \frac{x_a}{y_b z_b} - \frac{x_b}{y_b z_b}}{2} \\ &\quad + \frac{\frac{2x_a}{y_a z_a} - \frac{2x_a}{y_b z_a} + \frac{x_b}{y_b z_a} - \frac{x_b}{y_b z_b}}{2} \\ &= \frac{\frac{x_a}{y_b z_a} - \frac{x_b}{y_b z_a} + \frac{x_a}{y_b z_b} - \frac{x_b}{y_b z_b}}{2} \\ &= \frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b} \quad \square\end{aligned}$$

Three-variable quotient with two-variable denominator and Z responsibility center (Model 6)

This proof is the same as the proof for the three-variable quotient with two-variable denominator and Y

responsibility centers case, with y and z (and respective responsibility center) switched.

Three-variable quotient with two-variable denominator and X and Y responsibility centers (Model 7)

$$\begin{aligned}
 \frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b} &= \Delta x \left(\left(\frac{1}{y} \right) \frac{1}{z_a} \right) + \left(\Delta \frac{1}{y} \right) \left(\bar{x} \frac{1}{z_a} \right) \\
 &\quad + \left(\Delta \frac{1}{z} \right) \left(x_b \frac{1}{y_b} \right) \\
 &= (x_a - x_b) \left(\frac{\left(\frac{1}{y_a} + \frac{1}{y_b} \right) \frac{1}{z_a}}{2} \right) \\
 &\quad + \left(\frac{1}{y_a} - \frac{1}{y_b} \right) \left(\frac{(x_a + x_b) \frac{1}{z_a}}{2} \right) \\
 &\quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \left(x_b \frac{1}{y_b} \right) \\
 &= \frac{\frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b} + \frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_a}}{2} \\
 &\quad + \frac{\frac{x_a}{y_a z_a} - \frac{x_a}{y_b z_a} + \frac{x_b}{y_a z_b a} - \frac{x_b}{y_b z_a}}{2} \\
 &\quad + \left(\frac{x_b}{y_b z_a} - \frac{x_b}{y_b z_b} \right) \\
 &= \frac{\frac{x_a}{y_a z_a} - \frac{x_b}{y_a z_a} + \frac{x_a}{y_b z_a} - \frac{x_b}{y_b z_a}}{2} \\
 &\quad + \frac{\frac{x_a}{y_a z_a} - \frac{x_a}{y_b z_a} + \frac{x_b}{y_a z_a} - \frac{x_b}{y_b z_a}}{2} \\
 &\quad + \frac{\frac{2x_b}{y_b z_a} - \frac{2x_b}{y_b z_b}}{2} \\
 &= \frac{\frac{2x_a}{y_a z_a} - \frac{2x_b}{y_b z_b}}{2} \\
 &= \frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b} \quad \square
 \end{aligned}$$

Three-variable quotient with two-variable denominator and X and Z responsibility centers (Model 8)

This proof is the same as the proof for the three-variable quotient with two-variable denominator and X and Y responsibility centers case, with y and z (and respective responsibility center) switched.

Three-variable quotient with two-variable denominator and Y and Z responsibility centers (Model 9)

$$\begin{aligned}
 \frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b} &= \Delta x \left(\frac{1}{y_b} \frac{1}{z_b} \right) + \left(\Delta \frac{1}{y} \right) \left(x_a \left(\frac{1}{z} \right) \right) \\
 &\quad + \left(\Delta \frac{1}{z} \right) \left(x_a \left(\frac{1}{y} \right) \right) \\
 &= (x_a - x_b) \left(\frac{1}{y_b} \frac{1}{z_b} \right) \\
 &\quad + \left(\frac{1}{y_a} - \frac{1}{y_b} \right) \left(x_a \frac{\left(\frac{1}{z_a} + \frac{1}{z_b} \right)}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 &\quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \left(x_a \frac{\left(\frac{1}{y_a} + \frac{1}{y_b} \right)}{2} \right) \\
 &= \frac{x_a}{y_b z_b} - \frac{x_b}{y_b z_b} + \frac{\frac{x_a}{y_a z_a} - \frac{x_a}{y_b z_a} + \frac{x_a}{y_a z_b} - \frac{x_a}{y_b z_b}}{2} \\
 &\quad + \frac{\frac{x_a}{y_a z_a} + \frac{x_a}{y_b z_a} - \frac{x_a}{y_a z_b} - \frac{x_a}{y_b z_b}}{2} \\
 &= \frac{\frac{2x_a}{y_b z_b} - \frac{2x_b}{y_b z_b} + \frac{x_a}{y_a z_a} - \frac{x_a}{y_b z_a} + \frac{x_a}{y_a z_b} - \frac{x_a}{y_b z_b}}{2} \\
 &\quad + \frac{\frac{x_a}{y_a z_a} + \frac{x_a}{y_b z_a} - \frac{x_a}{y_a z_b} - \frac{x_a}{y_b z_b}}{2} \\
 &= \frac{\frac{2x_a}{y_a z_a} - \frac{2x_b}{y_b z_b}}{2} \\
 &= \frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b} \quad \square
 \end{aligned}$$

Three-variable quotient with two-variable denominator and X, Y and Z (or no) responsibility centers (Model 10)

$$\begin{aligned}
 \frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b} &= \Delta x \left(\left(\frac{1}{y} \right) \left(\frac{1}{z} \right) + \frac{\Delta \frac{1}{y} \Delta \frac{1}{z}}{12} \right) \\
 &\quad + \left(\Delta \frac{1}{y} \right) \left(\bar{x} \left(\frac{1}{z} \right) + \frac{\Delta x \Delta \frac{1}{z}}{12} \right) \\
 &\quad + \left(\Delta \frac{1}{z} \right) \left(\bar{x} \left(\frac{1}{y} \right) + \frac{\Delta x \Delta \frac{1}{y}}{12} \right) \\
 &= (x_a - x_b) \left(\frac{\left(\frac{1}{y_a} + \frac{1}{y_b} \right) \left(\frac{1}{z_a} + \frac{1}{z_b} \right)}{2} \right) \\
 &\quad + \frac{\left(\frac{1}{y_a} - \frac{1}{y_b} \right) \left(\frac{1}{z_a} - \frac{1}{z_b} \right)}{12} \\
 &\quad + \left(\frac{1}{y_a} - \frac{1}{y_b} \right) \left(\frac{(x_a + x_b) \left(\frac{1}{z_a} + \frac{1}{z_b} \right)}{2} \right) \\
 &\quad + \frac{(x_a - x_b) \left(\frac{1}{z_a} - \frac{1}{z_b} \right)}{12} \\
 &\quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \left(\frac{(x_a + x_b) \left(\frac{1}{y_a} + \frac{1}{y_b} \right)}{2} \right) \\
 &\quad + \frac{(x_a - x_b) \left(\frac{1}{y_a} - \frac{1}{y_b} \right)}{12} \\
 &= (x_a - x_b) + \left(\frac{1}{y_a} - \frac{1}{y_b} \right)
 \end{aligned}$$

$$\begin{aligned}
& \left(\frac{\frac{x_a}{z_a} + \frac{x_b}{z_a} + \frac{x_a}{z_b} + \frac{x_b}{z_b}}{4} + \frac{\frac{x_a}{z_a} - \frac{x_b}{z_a} - \frac{x_a}{z_b} + \frac{x_b}{z_b}}{12} \right) \\
& \quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \\
& \left(\frac{\frac{x_a}{y_a} + \frac{x_b}{y_a} + \frac{x_a}{y_b} + \frac{x_b}{y_b}}{4} + \frac{\frac{x_a}{y_a} - \frac{x_b}{y_a} - \frac{x_a}{y_b} + \frac{x_b}{y_b}}{12} \right) \\
& = \left(\frac{x_a - x_b}{12} \right) \left(\frac{\frac{3}{y_a z_a} + \frac{3}{y_b z_a} + \frac{3}{y_a z_b} + \frac{3}{y_b z_b}}{1} + \frac{1}{y_a z_a} - \frac{1}{y_b z_a} - \frac{1}{y_a z_b} + \frac{1}{y_b z_b} \right) \\
& \quad + \left(\frac{1}{y_a} - \frac{1}{y_b} \right) \left(\frac{\frac{3x_a}{z_a} + \frac{3x_b}{z_a} + \frac{3x_a}{z_b} + \frac{3x_b}{z_b}}{1} + \frac{x_a}{z_a} - \frac{x_b}{z_a} - \frac{x_a}{z_b} + \frac{x_b}{z_b} \right) \\
& \quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \left(\frac{\frac{3x_a}{y_a} + \frac{3x_b}{y_a} + \frac{3x_a}{y_b} + \frac{3x_b}{y_b}}{1} + \frac{x_a}{y_a} - \frac{x_b}{y_a} - \frac{x_a}{y_b} + \frac{x_b}{y_b} \right) \\
& = (x_a - x_b) \left(\frac{\frac{4}{y_a z_a} + \frac{2}{y_b z_a} + \frac{2}{y_a z_b} + \frac{4}{y_b z_b}}{12} \right) \\
& \quad + \left(\frac{1}{y_a} - \frac{1}{y_b} \right) \left(\frac{\frac{4x_a}{z_a} + \frac{2x_b}{z_a} + \frac{2x_a}{z_b} + \frac{4x_b}{z_b}}{12} \right) \\
& \quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \left(\frac{\frac{4x_a}{y_a} + \frac{2x_b}{y_a} + \frac{2x_a}{y_b} + \frac{4x_b}{y_b}}{12} \right) \\
& = \frac{1}{12} \left(\begin{aligned} & \frac{\frac{4x_a}{y_a z_a} + \frac{2x_a}{y_b z_a} + \frac{2x_a}{y_a z_b} + \frac{4x_a}{y_b z_b}}{4x_b} - \frac{\frac{4x_b}{y_a z_a} + \frac{2x_b}{y_b z_a} + \frac{2x_b}{y_a z_b} + \frac{4x_b}{y_b z_b}}{4x_a} \\ & + \frac{\frac{4x_a}{y_a z_a} + \frac{2x_b}{y_a z_a} + \frac{2x_a}{y_a z_b} + \frac{4x_b}{y_a z_b}}{4x_a} - \frac{\frac{4x_b}{y_a z_a} + \frac{2x_b}{y_a z_a} + \frac{2x_a}{y_a z_b} + \frac{4x_b}{y_a z_b}}{4x_b} \\ & + \frac{\frac{4x_a}{y_b z_a} + \frac{2x_b}{y_b z_a} + \frac{2x_a}{y_b z_b} + \frac{4x_b}{y_b z_b}}{4x_a} - \frac{\frac{4x_b}{y_b z_a} + \frac{2x_b}{y_b z_a} + \frac{2x_a}{y_b z_b} + \frac{4x_b}{y_b z_b}}{4x_b} \\ & + \frac{\frac{4x_a}{y_a z_a} + \frac{2x_b}{y_a z_a} + \frac{2x_a}{y_b z_a} + \frac{4x_b}{y_b z_a}}{4x_a} - \frac{\frac{4x_b}{y_a z_a} + \frac{2x_b}{y_a z_a} + \frac{2x_a}{y_b z_a} + \frac{4x_b}{y_b z_a}}{4x_b} \\ & - \frac{\frac{4x_a}{y_a z_b} + \frac{2x_b}{y_a z_b} + \frac{2x_a}{y_b z_b} + \frac{4x_b}{y_b z_b}}{4x_b} - \frac{\frac{4x_b}{y_a z_b} + \frac{2x_b}{y_a z_b} + \frac{2x_a}{y_b z_b} + \frac{4x_b}{y_b z_b}}{4x_a} \end{aligned} \right) \\
& = \frac{12x_a}{y_a z_a} - \frac{12x_b}{y_b z_b} \\
& = \frac{x_a}{y_a z_a} - \frac{x_b}{y_b z_b} \quad \square
\end{aligned}$$

Three-variable quotient with two-variable numerator and X responsibility center (Model 11)

$$\begin{aligned}
\frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b} &= \Delta x y_a \frac{1}{z_a} + \Delta y x_b \left(\frac{1}{z} \right) + \left(\Delta \frac{1}{z} \right) x_b \bar{y} \\
&= (x_a - x_b) \left(y_a \frac{1}{z_a} \right)
\end{aligned}$$

$$\begin{aligned}
& + (y_a - y_b) \left(x_b \frac{\left(\frac{1}{z_a} + \frac{1}{z_b} \right)}{2} \right) \\
& \quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \left(x_b \frac{(y_a + y_b)}{2} \right) \\
& = \frac{x_a y_a}{z_a} - \frac{x_b y_a}{z_a} + \frac{\frac{x_b y_a}{z_a} + \frac{x_b y_a}{z_b} - \frac{x_b y_b}{z_a} - \frac{x_b y_b}{z_b}}{2} \\
& \quad + \frac{\frac{x_b y_a}{z_a} - \frac{x_b y_a}{z_b} + \frac{x_b y_b}{z_a} - \frac{x_b y_b}{z_b}}{2} \\
& = \frac{\frac{2x_a y_a}{z_a} - \frac{2x_b y_a}{z_a}}{2} \\
& \quad + \frac{\frac{x_b y_a}{z_a} + \frac{x_b y_a}{z_b} - \frac{x_b y_b}{z_a} - \frac{x_b y_b}{z_b}}{2} \\
& \quad + \frac{\frac{x_b y_a}{z_a} - \frac{x_b y_a}{z_b} + \frac{x_b y_b}{z_a} - \frac{x_b y_b}{z_b}}{2} \\
& = \frac{\frac{2x_a y_a}{z_a} - \frac{2x_b y_b}{z_b}}{2} \\
& = \frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b} \quad \square
\end{aligned}$$

Three-variable quotient with two-variable numerator and Y responsibility center (Model 12)

This proof is the same as the proof for the three-variable quotient with two-variable numerator and X responsibility center case, with x and y (and respective responsibility center) switched.

Three-variable quotient with two-variable numerator and Z responsibility center (Model 13)

$$\begin{aligned}
\frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b} &= \Delta x \bar{y} \frac{1}{z_b} + \Delta y \bar{x} \frac{1}{z_b} + \left(\Delta \frac{1}{z} \right) x_a y_a \\
&= (x_a - x_b) \left(\frac{(y_a + y_b)}{2} \frac{1}{z_b} \right) \\
& \quad + (y_a - y_b) \left(\frac{(x_a + x_b)}{2} \frac{1}{z_b} \right) \\
& \quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) (x_a y_a) \\
& = \frac{\frac{x_a y_a}{z_b} + \frac{x_a y_b}{z_b} - \frac{x_b y_a}{z_b} - \frac{x_b y_b}{z_b}}{2} \\
& \quad + \frac{\frac{x_a y_a}{z_b} - \frac{x_a y_b}{z_b} + \frac{x_b y_a}{z_b} - \frac{x_b y_b}{z_b}}{2} \\
& \quad + \frac{\frac{x_a y_a}{z_a} - \frac{x_a y_a}{z_b}}{2} \\
& = \frac{\frac{x_a y_a}{z_b} + \frac{x_a y_b}{z_b} - \frac{x_b y_a}{z_b} - \frac{x_b y_b}{z_b}}{2} \\
& = \frac{\frac{2x_a y_a}{z_a} - \frac{2x_b y_b}{z_b}}{2} \\
& \quad + \frac{\frac{x_a y_a}{z_b} - \frac{x_a y_b}{z_b} + \frac{x_b y_a}{z_b} - \frac{x_b y_b}{z_b}}{2}
\end{aligned}$$

$$= \frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b} + \frac{\frac{2x_a y_a}{z_a} - \frac{2x_b y_b}{z_b}}{2} \quad \square$$

Three-variable quotient with two-variable numerator and X and Y responsibility centers (Model 14)

$$\begin{aligned} \frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b} &= \Delta x \bar{y} \frac{1}{z_a} + \Delta y \bar{x} \frac{1}{z_a} + \left(\Delta \frac{1}{z} \right) x_b y_b \\ &= (x_a - x_b) \left(\frac{(y_a + y_b) \frac{1}{z_a}}{2} \right) \\ &\quad + (y_a - y_b) \left(\frac{(x_a + x_b) \frac{1}{z_a}}{2} \right) \\ &\quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) (x_b y_b) \\ &= \frac{\frac{x_a y_a}{z_a} - \frac{x_b y_a}{z_a} + \frac{x_a y_b}{z_a} - \frac{x_b y_b}{z_a}}{\frac{2}{z_a} + \frac{x_b y_a}{z_a} + \frac{x_b y_a}{z_a} - \frac{x_a y_b}{z_a} - \frac{x_b y_b}{z_a}} \\ &\quad + \frac{\frac{2x_a y_a}{z_a} - \frac{2x_b y_b}{z_b}}{2} \\ &= \frac{\frac{x_a y_a}{z_a} - \frac{x_b y_a}{z_a} + \frac{x_a y_b}{z_a} - \frac{x_b y_b}{z_a}}{\frac{2}{z_a} + \frac{x_b y_a}{z_a} + \frac{x_b y_a}{z_a} - \frac{x_a y_b}{z_a} - \frac{x_b y_b}{z_a}} \\ &\quad + \frac{\frac{2x_a y_a}{z_a} - \frac{2x_b y_b}{z_b}}{2} \\ &= \frac{\frac{2x_a y_a}{z_a} - \frac{2x_b y_b}{z_b}}{2} \\ &= \frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b} \quad \square \end{aligned}$$

Three-variable quotient with two-variable numerator and X and Z responsibility centers (Model 15)

$$\begin{aligned} \frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b} &= \Delta x y_a \left(\frac{1}{z} \right) + \Delta y x_b \frac{1}{z_b} + \left(\Delta \frac{1}{z} \right) \bar{x} y_a \\ &= (x_a - x_b) \left(y_a \frac{\left(\frac{1}{z_a} + \frac{1}{z_b} \right)}{2} \right) \\ &\quad + (y_a - y_b) \left(x_b \frac{1}{z_b} \right) \\ &\quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \left(\frac{(x_a + x_b)}{2} y_a \right) \\ &= \frac{\frac{x_a y_a}{z_a} - \frac{x_b y_a}{z_a} + \frac{x_a y_b}{z_b} - \frac{x_b y_b}{z_b} + \frac{x_b y_a}{z_b}}{\frac{2}{z_b} + \frac{x_a y_a}{z_a} + \frac{x_b y_a}{z_a} - \frac{x_a y_b}{z_b} - \frac{x_b y_b}{z_b}} \\ &\quad - \frac{x_b y_b}{z_b} + \frac{\frac{2x_a y_a}{z_a} - \frac{2x_b y_b}{z_b}}{2} \\ &= \frac{\frac{x_a y_a}{z_a} - \frac{x_b y_a}{z_a} + \frac{x_a y_b}{z_b} - \frac{x_b y_b}{z_b}}{2} \\ &\quad + \frac{\frac{2x_a y_a}{z_a} - \frac{2x_b y_b}{z_b}}{2} \end{aligned}$$

$$\begin{aligned} &\quad + \frac{\frac{x_a y_a}{z_a} + \frac{x_b y_a}{z_a} - \frac{x_a y_b}{z_b} - \frac{x_b y_b}{z_b}}{2} \\ &= \frac{\frac{2x_a y_a}{z_a} - \frac{2x_b y_b}{z_b}}{2} \\ &= \frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b} \quad \square \end{aligned}$$

Three-variable quotient with two-variable numerator and Y and Z responsibility centers (Model 16)

This proof is the same as the proof for the three-variable quotient with two-variable numerator and X and Z responsibility centers case, with x and y (and respective responsibility center) switched.

Three-variable quotient with two-variable numerator and X, Y and Z (or no) responsibility centers (Model 17)

$$\begin{aligned} \frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b} &= \Delta x \left(\bar{y} \left(\frac{1}{z} \right) + \frac{\Delta y \Delta \frac{1}{z}}{12} \right) \\ &\quad + \Delta y \left(\bar{x} \left(\frac{1}{z} \right) + \frac{\Delta x \Delta \frac{1}{z}}{12} \right) + \Delta \frac{1}{z} \left(\bar{x} \bar{y} + \frac{\Delta x \Delta y}{12} \right) \\ &= (x_a - x_b) \\ &\quad \left(\frac{(y_a + y_b) \left(\frac{1}{z_a} + \frac{1}{z_b} \right)}{2} + \frac{(y_a - y_b) \left(\frac{1}{z_a} - \frac{1}{z_b} \right)}{12} \right) \\ &\quad + (y_a - y_b) \\ &\quad \left(\frac{(x_a + x_b) \left(\frac{1}{z_a} + \frac{1}{z_b} \right)}{2} + \frac{(x_a - x_b) \left(\frac{1}{z_a} - \frac{1}{z_b} \right)}{12} \right) \\ &\quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \\ &\quad \left(\frac{(x_a + x_b) (y_a + y_b)}{2} + \frac{(x_a - x_b) (y_a - y_b)}{12} \right) \\ &= (x_a - x_b) \\ &\quad \left(\frac{\frac{y_a}{z_a} + \frac{y_b}{z_a} + \frac{y_a}{z_b} + \frac{y_b}{z_b}}{4} + \frac{\frac{y_a}{z_a} - \frac{y_b}{z_a} - \frac{y_a}{z_b} + \frac{y_b}{z_b}}{12} \right) \\ &\quad + (y_a - y_b) \\ &\quad \left(\frac{\frac{x_a}{z_a} + \frac{x_b}{z_a} + \frac{x_a}{z_b} + \frac{x_b}{z_b}}{4} + \frac{\frac{x_a}{z_a} - \frac{x_b}{z_a} - \frac{x_a}{z_b} + \frac{x_b}{z_b}}{12} \right) \\ &\quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \\ &\quad \left(\frac{x_a y_a + x_b y_a + x_a y_b + x_b y_b}{4} \right. \\ &\quad \left. + \frac{x_a y_a - x_b y_a - x_a y_b + x_b y_b}{12} \right) \\ &= (x_a - x_b) \\ &\quad \left(\frac{\frac{3y_a}{z_a} + \frac{3y_b}{z_a} + \frac{3y_a}{z_b} + \frac{3y_b}{z_b} + \frac{y_a}{z_a} - \frac{y_b}{z_a} - \frac{y_a}{z_b} + \frac{y_b}{z_b}}{12} \right) \\ &\quad + (y_a - y_b) \end{aligned}$$

$$\begin{aligned}
& \left(\frac{\frac{3x_a}{z_a} + \frac{3x_b}{z_a} + \frac{3x_a}{z_b} + \frac{3x_b}{z_b} + \frac{x_a}{z_a} - \frac{x_b}{z_a} - \frac{x_a}{z_b} + \frac{x_b}{z_b}}{12} \right) \\
& \quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \\
& \quad \left(\frac{3x_a y_a + 3x_b y_a + 3x_a y_b + 3x_b y_b}{12} + \frac{x_a y_a - x_b y_a - x_a y_b + x_b y_b}{12} \right) \\
& = (x_a - x_b) \left(\frac{\frac{4y_a}{z_a} + \frac{2y_b}{z_a} + \frac{2y_a}{z_b} + \frac{4y_b}{z_b}}{12} \right) \\
& \quad + (y_a - y_b) \left(\frac{\frac{4x_a}{z_a} + \frac{2x_b}{z_a} + \frac{2x_a}{z_b} + \frac{4x_b}{z_b}}{12} \right) \\
& \quad + \left(\frac{1}{z_a} - \frac{1}{z_b} \right) \\
& \quad \left(\frac{4x_a y_a + 2x_b y_a + 2x_a y_b + 4x_b y_b}{12} \right) \\
& = \frac{1}{12} \left(\begin{aligned} & \frac{4x_a y_a}{z_a} + \frac{2x_a y_b}{z_a} + \frac{2x_a y_a}{z_b} + \frac{4x_a y_b}{z_b} \\ & - \frac{4x_b y_a}{z_a} - \frac{2x_b y_b}{z_a} - \frac{2x_b y_a}{z_b} - \frac{4x_b y_b}{z_b} \\ & + \frac{4x_a y_a}{z_a} + \frac{2x_b y_a}{z_a} + \frac{2x_a y_a}{z_b} + \frac{4x_b y_a}{z_b} \\ & - \frac{4x_a y_b}{z_a} - \frac{2x_b y_b}{z_a} - \frac{2x_a y_b}{z_b} - \frac{4x_b y_b}{z_b} \\ & + \frac{4x_a y_a}{z_a} + \frac{2x_b y_a}{z_a} + \frac{2x_a y_b}{z_a} + \frac{4x_b y_b}{z_a} \\ & - \frac{4x_a y_a}{z_b} - \frac{2x_b y_a}{z_b} - \frac{2x_a y_b}{z_b} - \frac{4x_b y_b}{z_b} \end{aligned} \right) \\
& = \frac{\frac{12x_a y_a}{z_a} - \frac{12x_b y_b}{z_b}}{12} \\
& = \frac{x_a y_a}{z_a} - \frac{x_b y_b}{z_b} \quad \square
\end{aligned}$$